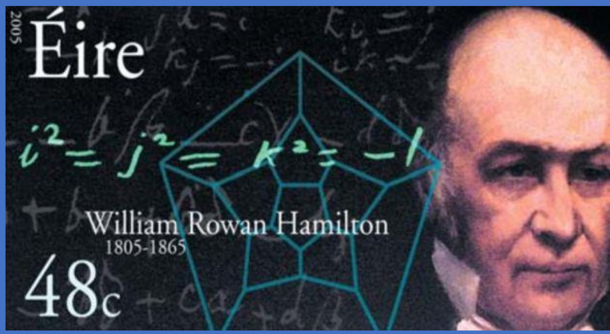


# MATHEMATICIANS IN OUR LIVES

With the support of

15-16 years olds

## SECTION 1 - WILLIAM ROWAN HAMILTON

1. Who is William Rowan Hamilton?

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2. What was his best known discovery?

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3. Where was the bridge that Hamilton carved equations into?

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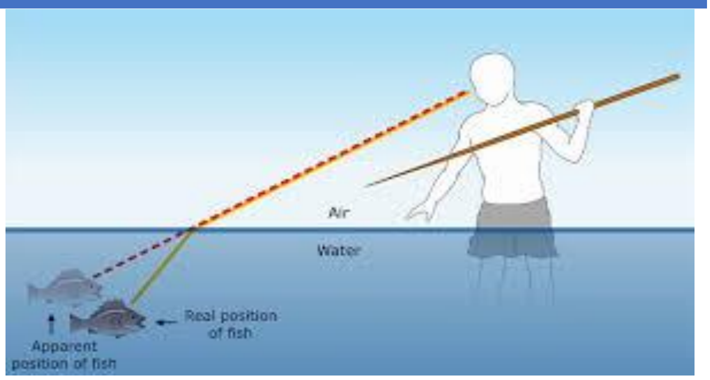
## SECTION 2 - OPTICS

4. What is refraction?

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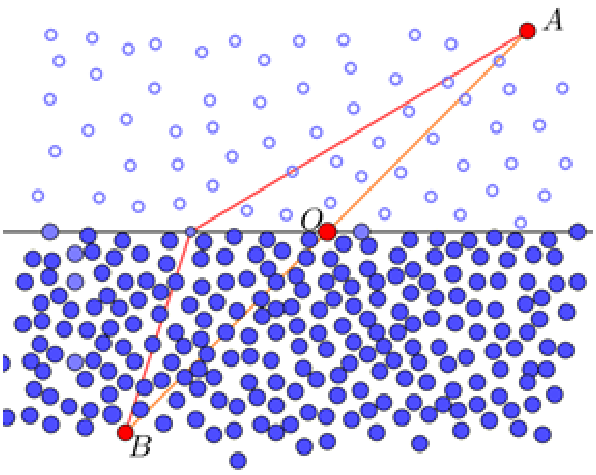
5. Normally, the fastest path between two points is a straight line - but not when you hit obstacles, which cause delays.

Explain why the longer red path in this figure could be faster than the straight path AOB.




6. What is the Minimum Principle?

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7. What causes refraction to happen?

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8. Why do you think the light travels along the fastest paths?

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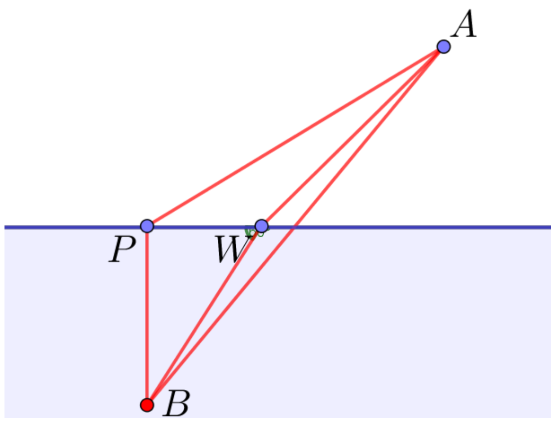
9. In the Light SuperWorld, light rays travel on any paths they like. Three rays called Mr Simples, Mrs Wiseman and James Bold, decide to go from a point $A$, found 100 meters above water, to a point $B$, found 100 meters below water. They are warned that travelling through water is slower, namely

- They can travel at a speed of 300 meters/second through air;
- But only 225 meters/second through water.

Mr Simples decides to take a straight line from A to B, a total distance of 255 meters.

James Bold decides to go as much as possible through air, so he travels 187 m to point P, found exactly above B on the surface of the water, and then from P straight down to B.

Mrs Wiseman Ray makes some calculation and decides to go about 141.5 meters through the air, heading straight for a point W on the water surface, and then travels about 115.5 meters through the water, from W to B.




Which Ray gets to the destination fastest? Can you intuitively explain why?

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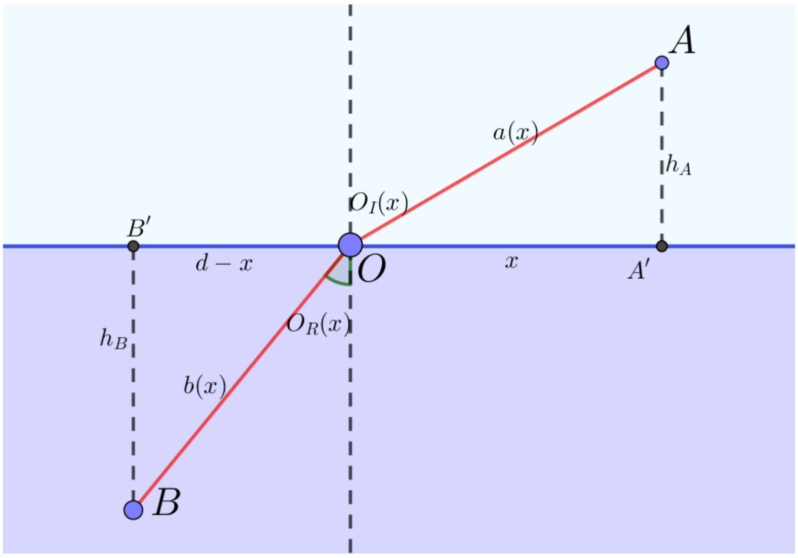
10. *Challenge: Where does the Refraction Formula come from?*




Based on the minimum principle, we can show that the fastest path happens exactly when the Refraction Formula holds. We will use the Refraction Formula in its form $\frac{\sin O_I}{v_I} - \frac{\sin O_R}{v_R} = 0$.

Fix a source *A* and a target *B.* Imagine sliding the point *O* along the border, so that the distance $x = |OA'|$ varies in between *0* and $d$ =$|A'B'|$. Notice all the quantities that change with $x$. They are denoted as functions of $x$ in the picture: $a(x)$, $b(x)$, $O_{I(x)}$, $O_R(x)$, $d-x$ as well as $t(x)$, the total time for the trip AOB.

Choose two points inside the segment $A'B'$ such that $x_1 > x_2$.

a) Check that the following equations are true:

$$\sin O_I(x_1) = \frac{x_1}{a(x_1)} > \frac{x_1 + x_2}{a(x_1) + a(x_2)} > \frac{x_2}{a(x_2)} = \sin O_I(x_2) \ ,$$

$$-\sin O_R(x_1) = -\frac{d-x_1}{b(x_1)} > -\frac{2d - x_1 - x_2}{b(x_1) + b(x_2)} > -\frac{d-x_2}{b(x_2)} = -\sin O_R(x_2)\,,$$

b) To compare $t(x_1)$ and $t(x_2)$, calculate the difference $t(x_1) - t(x_2)$ and show that

$$\frac{t(x_1) - t(x_2)}{x_1 - x_2} < \left(\frac{\sin O_I(x_1)}{v_I} - \frac{\sin O_R(x_1)}{v_R}\right).$$

Hence if $\frac{\sin O_I(x_1)}{v_I} - \frac{\sin O_R(x_1)}{v_R} = 0$ (Refraction Formula), we have $t(x_1) < t(x_2)$ (The time of the trip is shorter if you pass through the point which satisfies the Refraction Formula).

*Hint*: Use the difference of two squares formula in the form

$$a(x_1) - a(x_2) = \frac{a^2(x_1) - a^2(x_2)}{a(x_1) + a(x_2)}.$$

c) Similarly, if $x_2 > x_1$, you can check $\left(\frac{\sin O_I(x_1)}{v_I} - \frac{\sin O_R(x_1)}{v_R}\right) < \frac{t(x_2) - t(x_1)}{x_2 - x_1}$.

From here you can conclude that $t(x_1)$ is shortest when $\frac{\sin O_I(x_1)}{v_I} - \frac{\sin O_R(x_1)}{v_R} = 0$.

## SECTION 3 - GRAPH THEORY

11. What was the game of the Seven Bridges of Königsberg?

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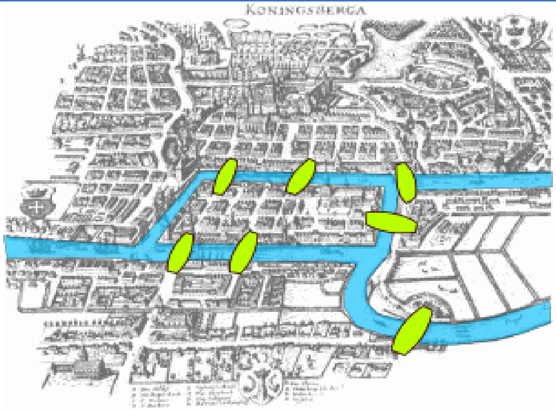


12. Why is it impossible to win the game of the Seven Bridges of Königsberg?

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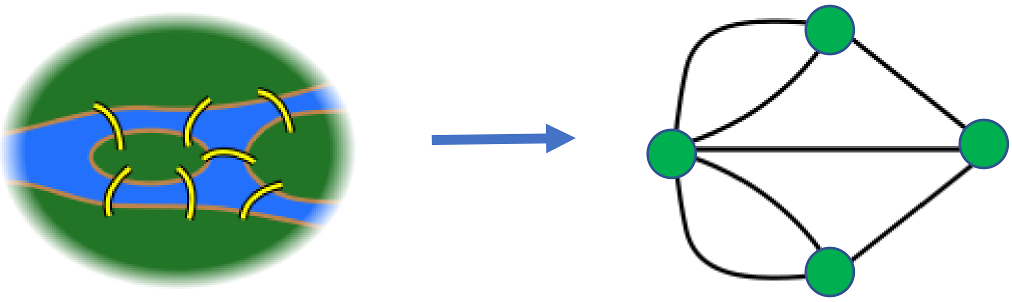

13. What is a graph in graph theory? Draw an example of a graph.

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14. Below are the 5 Platonic solids and their planar graphs. The graphs are obtained by stretching the bottom face of each Platonic solid to make is much larger than all others, and then squashing the solid from the top until it's flattened and it fits inside its base. The graphs are not in the right order.
    a) What special properties do all the Platonic Solids share?
    b) Connect each graph by an arrow to the Platonic Solid it corresponds to.
    c) Find Hamiltonian circuits (cycles) on each of the 5 graphs.

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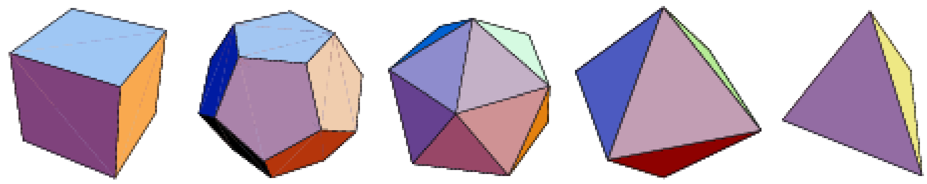
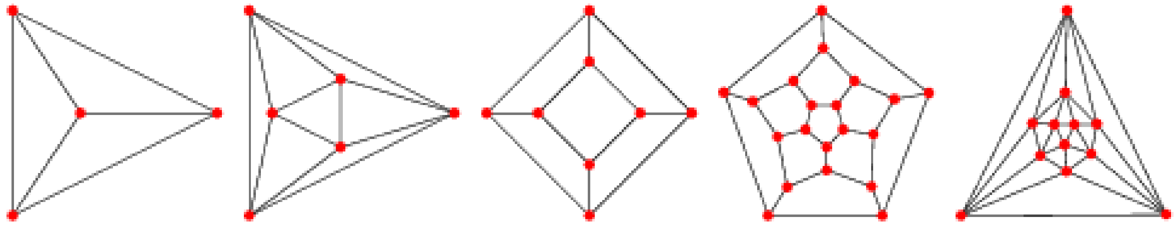
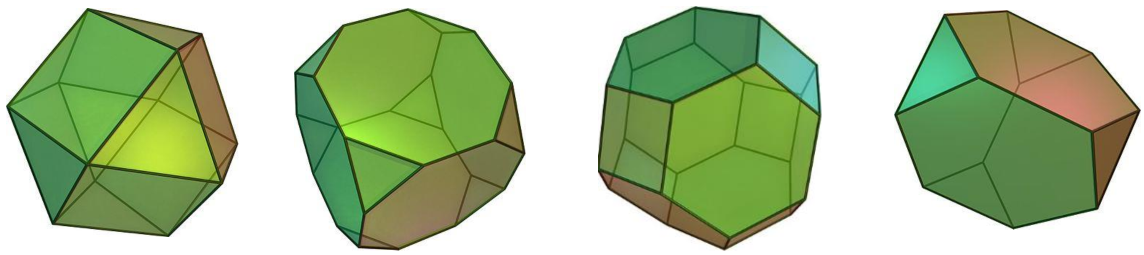


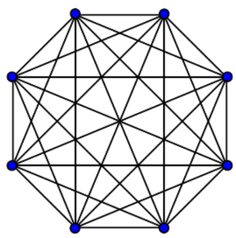


15. Try the same with some of the following Archimedean solids. You will first need to stretch one of their larger faces and then squash them to planar graphs.



Cuboctahedron | Truncated cube | Truncated octahedral | Truncated tetrahedral

16. A *complete graph* is a simple graph where every node is connected with every other node by exactly one edge. In the diagram you can see a complete graph with 8 nodes.



   a) Draw diagrams for simple graphs with:
   (i) 2 nodes; (ii) 3 nodes; (iii) 4 nodes; (iv) 5 nodes (v) 6 nodes (vi) 7 nodes

   and count their edges. Find a formula for the number of edges of a complete graph with $n$ nodes.

   b) Every complete graph with at least 3 nodes is Hamiltonian. For example, a complete graph with 3 nodes has 1 Hamiltonian cycle, and a complete graph with 4 nodes has 3 Hamiltonian cycles (find them all). Find a formula for the total number of Hamiltonian cycles that can be found in a complete graph with $n$ nodes.

17. Use reflection to explain why multiplying two negative numbers gives you a positive number.

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18. Work out how to reflect around a number other than $0$. Let's say that we have a number $A$ on the line and another number $x$. Write an equation for the reflection of $x$ through $A$. It should be an expression in $x$ and $A$.

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19. Show that the two numbers whose sum is 10 and product is 40, would be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ if we allowed for square roots of a negative numbers.

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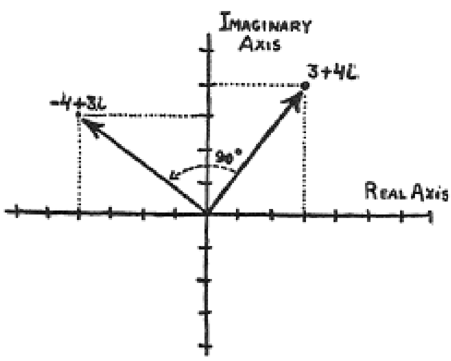
20. Check the position of the point $i \cdot (3 + 4i) = -4 + 3i$ in the figure above. Check that the ray of $(-4 + 3i)$ is perpendicular on the ray of $(3 + 4i)$.

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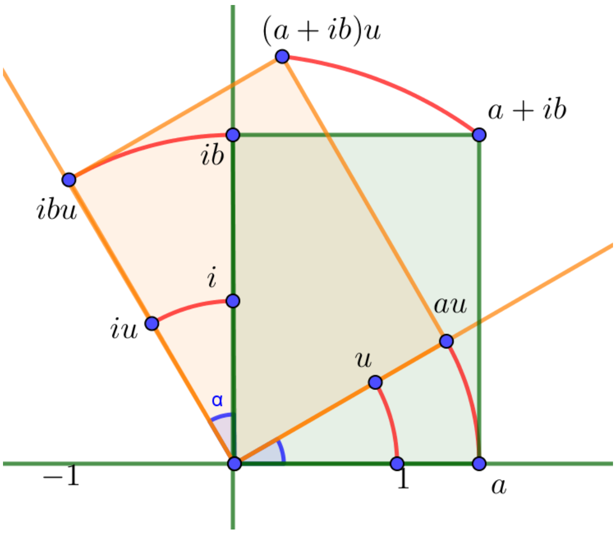
21. *Products by complex numbers can describe rotations by all angles*:




Let's take a complex number $u$ obtained by rotating the number 1 from the horizontal axis by $\alpha$ degrees. Such a number is written as $u = \cos\alpha + i\sin\alpha$ .

What does the picture here tell us about $au$ when compared with $a$ ? How about $ibu$ when compared with $ib$ ?

What does the picture here tell us about $(a + ib)u$ when compared with $a + ib$ ?

How would you describe $\cdot\, u$ geometrically?

22. *The Treasure-Hunting Puzzle* *(from "One, two, three… infinity" by George Gamow)*:
If you still feel a veil of mystery surrounding imaginary numbers you will probably be able to disperse it by working out how to find some buried treasure using complex numbers.

There was a young and adventurous man who found among his great-grandfather's papers a piece of parchment that revealed the location of a hidden treasure. The instructions read: "Sail to _________ North latitude and __________ West longitude where you will find a deserted island. There on a large Meadow stand a lonely oak and a lonely pine. There you will see also an old gallows on which we once hanged traitors. Start from the gallows and walk to the oak counting thy steps. At the oak you must turn right by a right angle and take the same number of steps. Put here a spike in the ground. Now you must return to the gallows and walk to the pine counting the steps. At the pine you must turn left by a right angle and see that you take the same number of steps, and put another spike into the ground. Dig halfway between the spikes; the treasure is there."

The instructions were quite clear and explicit, so our young man chartered a ship and sailed to the South Seas. He found the island, the field, the oak and the pine, but to his great sorrow the gallows was gone. Too long a time had passed since the document had been written; rain and

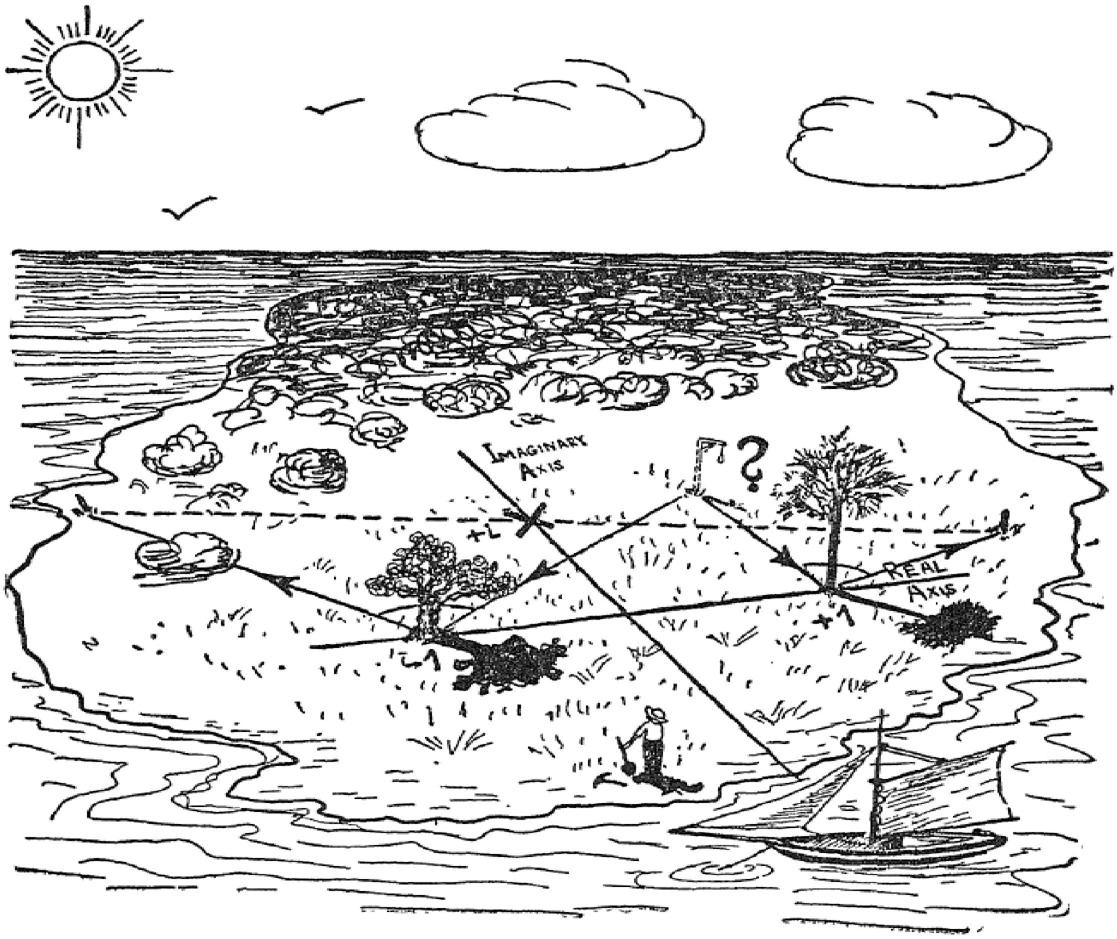



FIGURE 11
Treasure hunt with imaginary numbers.

sun and wind had disintegrated the wood and returned it to the soil, leaving no trace even of the place where it once had stood. Our adventurous young man fell into despair, then in an angry frenzy began to dig at random all over the field. But all his efforts were in vain; the island was too big! So he sailed back with empty hands. And the treasure is probably still there.

> A sad story, but what is sadder still is the fact that the fellow might have had the treasure, if only he had known a bit about mathematics, and specifically the use of imaginary numbers. Let us see if we can find the treasure for him, even though it is too late to do him any good.

Consider the island as a plane of complex numbers; draw the real (horizontal) axis through the base of the two trees, and another axis (the imaginary one) at right angles to the first, through a point half way between the trees. Taking $\frac{1}{2}$ the distance between the trees as our unit of length, we can say that the oak is located at the point $-1$ on the real axis, and the pine at the point $+1$. We do not know where the gallows was so let us denote its hypothetical location by the Greek letter $\Gamma$ (capital gamma), which even looks like a gallows. Since the gallows was not necessarily on one of the two axes, $\Gamma$ must be considered as a complex number: $\Gamma = a + bi$.
Now let us do some calculations remembering the rules of imaginary multiplication as stated above. If the oak is at $-1$, the gallows is at $\Gamma$, then the walk from the oak to the gallows may be written as:

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Now to rotate it counter-clockwise by $90°$, we multiply by $i$ and get:

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This describes the trip from the oak to the 1st spike.
Since the oak is at point $-1$, the trip from the origin O to the first spike is:

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Similarly the walk from the pine at 1 to the gallows at $\Gamma$ is written as

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We rotate it clockwise around the pine by multiplying by $-i$

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and then calculate the trip from O to the second spike as:

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Since the treasure is halfway between the spikes, we must now find one half the sum of the two above complex numbers. We get:

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We now see that the unknown position of the gallows denoted by $\Gamma$ fell out of our calculations somewhere along the way, and that, regardless of where the gallows stood, the treasure must be located at the point ____________________________________________________________

And so, if our adventurous young man could have done this bit of mathematics, he would not have needed to dig up the entire island, but would have looked for the treasure at the point indicated by the cross in Figure 11, and there would have found the treasure.

23. Take two lines l and k meeting at O and with an angle $\alpha$ between them. Reflect a point P through l and then k successively to get $P'$ and then $P''$.
Show that $|OP''| = |OP'| = |OP|$ and that $\angle P''OP = 2\alpha$ no matter which $P$ you choose.

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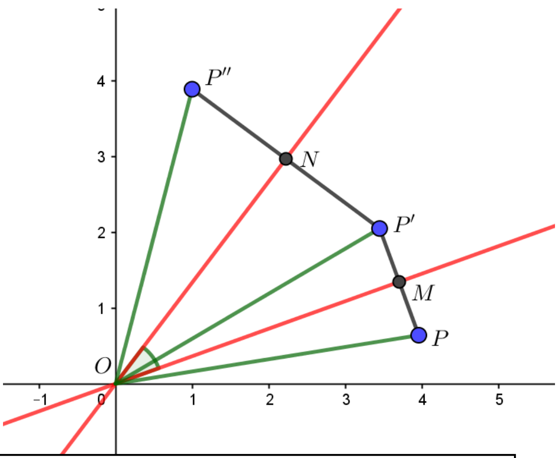



Two successive reflections across two lines through O amounts to a rotation by double the angle between the two lines

24. Fill in the table with the correct Hamilton products. For each box, its row represents the first number in the product, while its column is the second number.

For example, I've placed $k$ in row $i$ and column $j$ because $ij = k$.

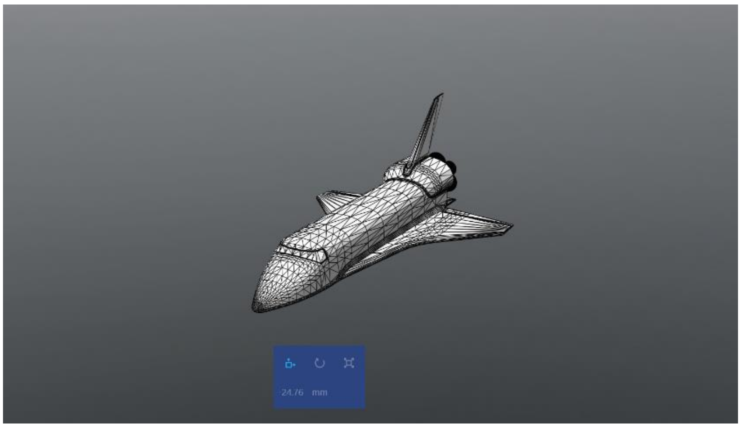
| × | **1** | ***i*** | ***j*** | ***k*** |
|---|---|---|---|---|
| **1** | | | | |
| ***i*** | | | $k$ | |
| ***j*** | | | | |
| ***k*** | | | | |

25. Try to multiply these out:

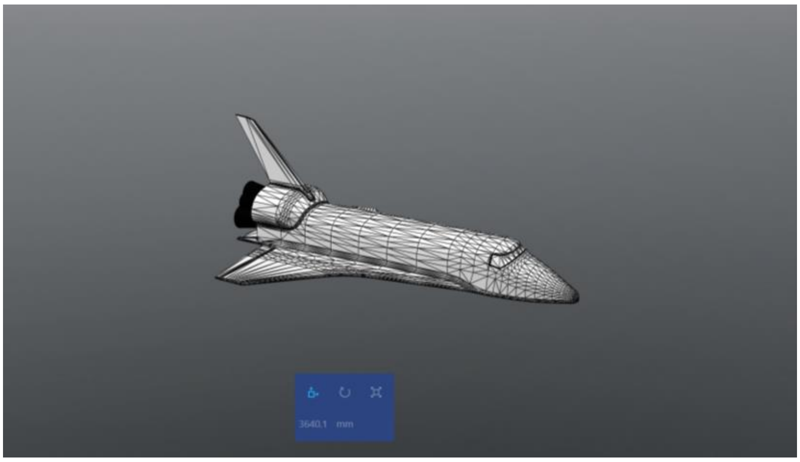
a) $(3\boldsymbol{j} + 4\boldsymbol{k})(6\boldsymbol{j} + 8\boldsymbol{k}) =$

b) $(1 + 3\boldsymbol{i} + 10\boldsymbol{j})(1 + 2\boldsymbol{k}) =$

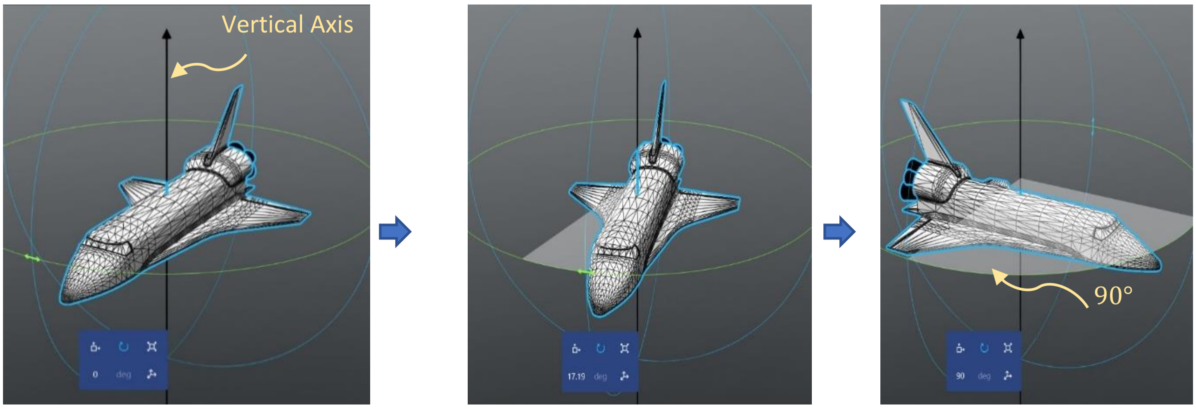
26. Imagine there are astronauts on a spacecraft.



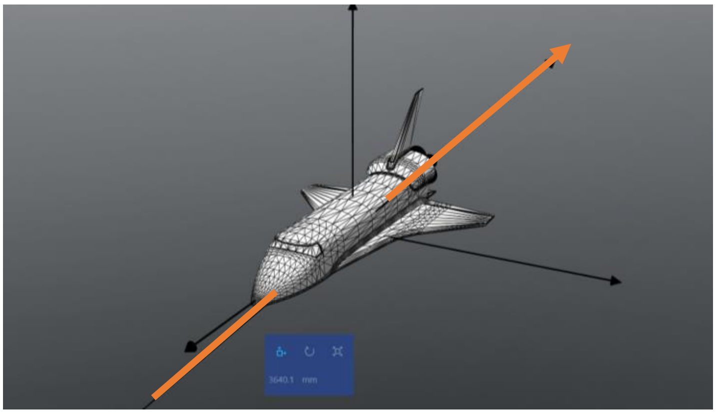
The astronauts are having a tanning competition on board and want to turn the shuttle by 90° to its left so that it faces the sun, like this:



How does the shuttle carry out this command (in terms of quaternions)?

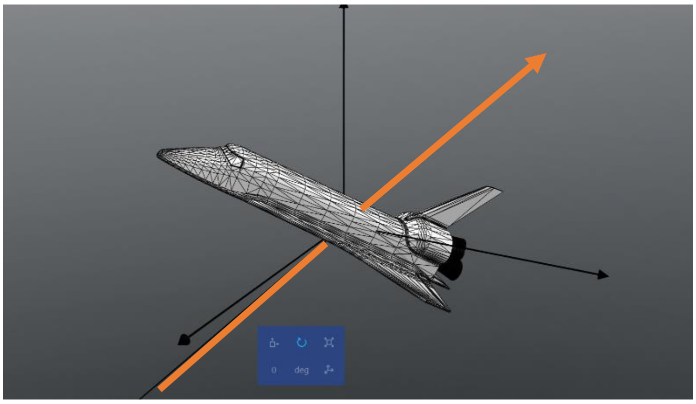



27. Let's try something a bit trickier this time. We will try to rotate clockwise by 45° around the following axis this time (maybe they've seen aliens or a space duck off to their right, or something):



This time, we'll use $\left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ when we talk about the axis. Can you fill in the boxes in the rotation quaternion?

$$\boldsymbol{q} = \square + \square\,\boldsymbol{i} + \square\,\boldsymbol{j} + \square\,\boldsymbol{k}$$



28. Calculate the products $iji$, $iki$ and $iii$.

Now take a number and surround it by $i$ and $i$, like this: $i(ai + bj + ck)i$. Plot your results in 3D space. What do you notice? Can you describe this operation as a geometric transformation (movement) of the point $p = ai + bj + ck$?

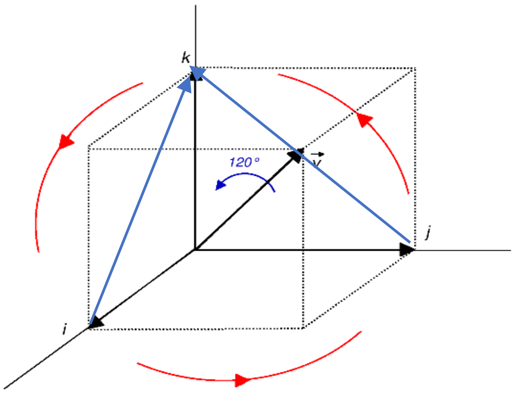
| |
|---|
| The product $upu$ = the reflection of $p$ across the plane through 0 and perpendicular to $u$. |

| |
|---|
| $vupuv$ = two successive reflections of $p$ across the planes through O perpendicular on $u$ and $v$.<br>= rotation by angle $2\alpha$ around the axis $n$ which is perpendicular on both $u$ and $v$. |

29. Consider the quaternions $u = \frac{1}{\sqrt{2}}(i - k)$ and $v = \frac{1}{\sqrt{2}}(j - k)$. The directions of these vectors are indicated by the blue arrows in the diagram.
Find the result of the transformation

$$p \to upu \to vupuv.$$

in the cases $p = i$, $p = j$ and $p = k$. Use this to explain why this transformation is the rotation by $120°$ around the diagonal of the cube – as shown in the picture.

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ADVANCED CHALLENGE: FOR MATHS CLUBS, MATHS CIRCLES OR MATHS PROJECTS

30. Multiply two quaternions $u = u_1 i + u_2 j + u_3 k$ and $v = v_1 i + v_2 j + v_3 k$.

Use $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$ as before to write the formula for $uv$ in the form $\boldsymbol{uv} = \square + \square\, \boldsymbol{i} + \square\, \boldsymbol{j} + \square\, \boldsymbol{k}$

After doing this thoroughly, you will get this formula:

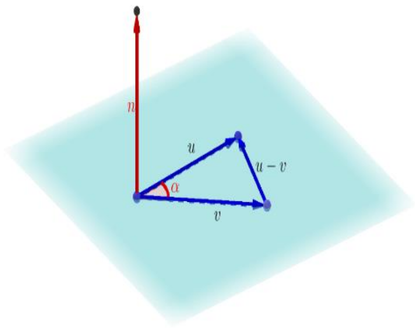
$$uv = -(u_1v_1 + u_2v_2 + u_3v_3) + (u_2v_3 - u_3v_2)\,\boldsymbol{i} + (u_3v_1 - u_1v_3)\,\boldsymbol{j} + (u_1v_2 - u_2v_1)\,\boldsymbol{k}.$$

Now here is a wonderful property of the quaternion product that Hamilton has long sought:

$$|uv| = |u| \cdot |v|$$

The length of a product is the product of lengths.

Check this formula using the coordinates.




31.

a) Apply the cosine formula in the triangle formed by the quaternions $u$ and $v$ with $u - v$, to check

$$\cos\alpha = u_1v_1 + u_2v_2 + u_3v_3.$$

Use the rule $\cos^2\alpha + \sin^2\alpha = 1$ to find

$$\sin\alpha = \sqrt{(u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2}$$

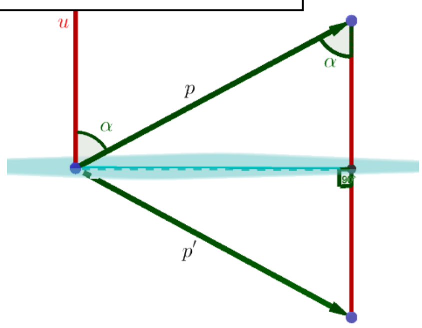
b) Define $n = \frac{1}{\sin\alpha}[(u_2v_3 - u_3v_2)\,\boldsymbol{i} + (u_3v_1 - u_1v_3)\,\boldsymbol{j} + (u_1v_2 - u_2v_1)\,\boldsymbol{k}]$.
Apply the formula (a) for the cosine of an angle between two vectors to show that $n$ is perpendicular to both $u$ and $v$.
*Hint*: You need to show that $\cos\,(angle\ between\ n\ and\ u) = 0$ and similarly for $n$ and $v$
You can substitute $(u_2v_3 - u_3v_2)\,\boldsymbol{i} + (u_3v_1 - u_1v_3)\,\boldsymbol{j} + (u_1v_2 - u_2v_1)\,\boldsymbol{k}$ for $\boldsymbol{n}$**,** since $n$ is just a rescaling of the former vector.

$$uv = -\cos\alpha + \sin\alpha \cdot \boldsymbol{n}$$

Where $\alpha$ is the angle between $u$ and $v$, while $\boldsymbol{n}$ is the unit vector perpendicular to both $u$ and $v$.

32. To prove the ***Reflection Formula****:*

$upu$ = the reflection of $p$ across the plane through 0 which is perpendicular to $u$.




In this exercise, you can work with a point $p$ on the Quaternion Sphere so $|p| = 1$.

a) Using the diagram here, show that the reflection of $p$ across the plane perpendicular on $u$ is the vector

$$p' = p - 2\cos\alpha \cdot u$$

b) $up + pu = -2\cos\alpha$ and

$$upu = p - 2\cos\alpha \cdot u = p'.$$